

Last time: Connected geometry with algebra (for vectors)

12.3: Dot Product

Goal: Connect algebra of vectors to their geometry via a new operation on vectors

Definition: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle, \vec{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3$

The dot product of \vec{u} and \vec{v} is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

(vector \cdot vector \mapsto scalar)

Q: Does $\vec{u} \cdot (\vec{v} \cdot \vec{w})$ make sense?

No! It doesn't make sense. $\vec{v} \cdot \vec{w}$ makes a scalar so this problem tries to take a vector dot a scalar.

$((\vec{u} \cdot \vec{v}) \vec{w})$ makes sense via scalar multiplication

Ex: $\vec{u} = \langle 1, 3, 5 \rangle, \vec{v} = \langle -3, 5, 7 \rangle$

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (1)(-3) + (3)(5) + (5)(7) \\ &= -3 + 15 + 35 = 47\end{aligned}$$

Theorem (Properties of dot product): (POD)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

$$1) \vec{v} \cdot \vec{v} = v_1 v_1 + v_2 v_2 + v_3 v_3 = v_1^2 + v_2^2 + v_3^2 = \|\vec{v}\|^2$$

$$2) \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = v_1 u_1 + v_2 u_2 + v_3 u_3 = \vec{v} \cdot \vec{u}$$

$$3) \vec{u} \cdot (\vec{v} + \vec{w}) = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \\ = u_1(v_1 + w_1) + u_2(v_2 + w_2) + u_3(v_3 + w_3)$$

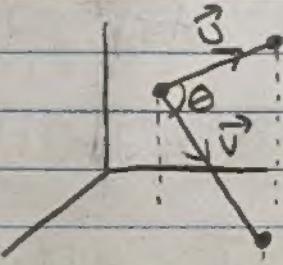
$$= (u_1 v_1 + u_2 v_2 + u_3 v_3) + (u_1 w_1 + u_2 w_2 + u_3 w_3) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\begin{aligned}
 4) \vec{u} \cdot (c\vec{v}) &= \langle u_1, u_2, u_3 \rangle \cdot \langle cv_1, cv_2, cv_3 \rangle \\
 &= u_1(cv_1) + u_2(cv_2) + u_3(cv_3) \\
 &= c(u_1v_1 + u_2v_2 + u_3v_3) = c(\vec{u} \cdot \vec{v})
 \end{aligned}$$

$$5) \vec{0} \cdot \vec{v} = 0 \quad \text{"zero property"}$$

Theorem (Geometric interpretation of Dot Product):

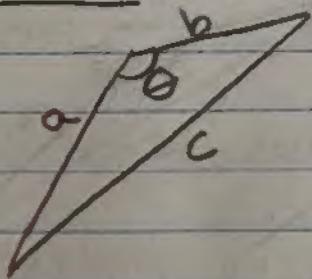
Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let θ be the angle between them



$$\text{Then } \vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos(\theta)$$

purely algebraic purely geometric

Recall: Law of cosines

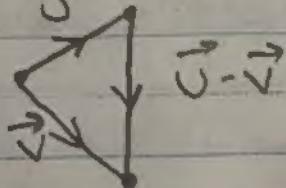


$$c^2 = a^2 + b^2 - 2ab \cos(\theta)$$

(generalization of the Pythagorean Theorem)

Proof: Consider $\vec{u} - \vec{v}$. Applying law of cosines

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos \theta$$



On the left side, we apply properties of the dot product.

$$\begin{aligned}
 |\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) && \text{(Part 1 POD)} \\
 &= (\vec{u} - \vec{v}) \cdot \vec{u} - (\vec{u} - \vec{v}) \cdot \vec{v} && \text{(Part 2 POD)} \\
 &= (\vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v}) - (\vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v}) && \text{(Distribution of)} \\
 &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} && \text{(Algebra) dot} \\
 &= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2\vec{u} \cdot \vec{v} && \text{(commutativity of dot)} \\
 &= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v} && \text{(Part 1 again)}
 \end{aligned}$$

Hence $(|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos(\theta))$

$$\begin{aligned}
 &= |\vec{u} - \vec{v}|^2 \\
 &= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}
 \end{aligned}$$

Finally, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

Corollary: Supposing \vec{u}, \vec{v} are both nonzero

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)$$

* \arccos
same as \cos^{-1}

Observation: The zero-vector has an undefined angle with all other vectors

So having an angle means no non-zero vectors

Corollary: If \vec{u} and \vec{v} are perpendicular (i.e. orthogonal), then $\vec{u} \cdot \vec{v} = 0$

Conversely, $\vec{u} \cdot \vec{v} = 0$ implies \vec{u} and \vec{v} are orthogonal

Orthogonal Projection

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^n$



To project \vec{u} orthogonally onto \vec{v} :

$$c\vec{v} \cdot (\vec{u} - c\vec{v}) = 0$$

$$\text{if } c(\vec{v} \cdot \vec{v}) - c^2(\vec{v} \cdot \vec{v}) = 0$$

$$c(\vec{v} \cdot \vec{v} - c|\vec{v}|^2) = 0$$

$$\text{either } c=0 \text{ or } \vec{v} \cdot \vec{v} - c|\vec{v}|^2 = 0$$

So assuming $|\vec{v}| \neq 0$ and $c \neq 0$

$$c = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}$$

Definition: The orthogonal projection of \vec{u} onto \vec{v} is:

$$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

$$= \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} \cdot \left(\frac{1}{|\vec{v}|} \vec{v} \right)$$

$$= \text{Comp}_{\vec{v}}(\vec{u}) \left(\frac{1}{|\vec{v}|} \vec{v} \right)$$

The scalar projection of \vec{u} onto \vec{v} is $\vec{u} \cdot \vec{v}$

$$\text{Comp}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

Direction Angles

Let $\vec{v} \in \mathbb{R}^3$

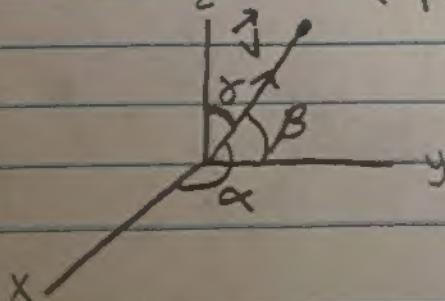
$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

The direction angles of \vec{v} are the angles \vec{v} makes with \vec{i} , \vec{j} , and \vec{k} .

$$\text{i.e. } \alpha = \arccos \left(\frac{\vec{v} \cdot \vec{i}}{|\vec{v}| |\vec{i}|} \right) = \arccos \left(\frac{v_1}{|\vec{v}|} \right)$$

$$\beta = \arccos \left(\frac{v_2}{|\vec{v}|} \right)$$

$$\gamma = \arccos \left(\frac{v_3}{|\vec{v}|} \right)$$



The direction angles determine the "would-be location" of \vec{v} on the unit sphere about the origin

Exercise: Show that any two of the direction angles of \vec{v} determine the third...